Topological expansion and boundary conditions

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Abstract:

In this article, we compute the topological expansion of all possible mixed-traces in a hermitian two matrix model. In other words we give a recipe to compute the number of discrete surfaces of given genus, carrying an Ising model, and with all possible given boundary conditions. The method is recursive, and amounts to recursively cutting surfaces along interfaces. The result is best represented in a diagrammatic way, and is thus rather simple to use.

1 Introduction

1.1 Counting surfaces with given boundary conditions

The problem of boundary conditions is a very important one in statistical mechanics, conformal field theory, string theory... (see for example [2, 16, 19] for recent developments). In this article we address the problem of counting configurations of an Ising model on a random lattice, with given boundary conditions. This problem can be equivalently stated as computing mixed traces expectation values in a 2-matrix model.

The 2-matrix model was introduced by Kazakov [17] as the Ising model on a random lattice. Its partition function reads:

$$Z = \int dM_1\, dM_2 \;\; \mathrm{e}^{-N\,\mathrm{Tr}\,[V_1(M_1)+V_2(M_2)-M_1 M_2]} \tag{1-1}$$

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$$\left\langle \operatorname{Tr} M_1^3 \operatorname{Tr} M_2^5 \right\rangle_c :=$$

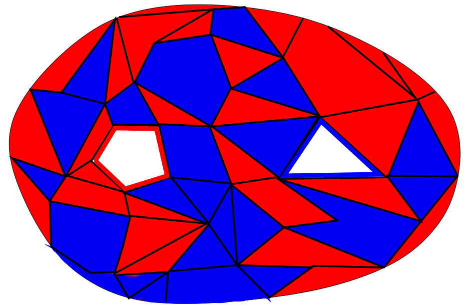


Figure 1: Example of surface generated by $\langle \operatorname{Tr} M_1^3 \operatorname{Tr} M_2^5 \rangle_c$, where one has associated the blue color to M_1 and the red color to M_2: it is a cylinder with one boundary of length 5 with red condition and one boundary of length 3 with blue condition.

where V_1 and V_2 are polynomials, and where the integral is a **formal** hermitian matrix integral (see for example [13] for a definition of formal integrals), i.e. it is computed by first expanding the exponential of the non-quadratic part of V_1 and V_2, and then exchanging the sums and integrals. A formal integral is thus a formal series whose general terms are moments of gaussian integrals [13].

It is well known from Wick's theorem that such a formal integral is a combinatorial generating function which enumerates discrete surfaces (also called maps in the combinatorists litterature) whose faces can have 2 possible colors 1 or 2, or let us say + or $-$, or blue or red.

The moments:

$$< \operatorname{Tr} M_1^l > \tag{1-2}$$

are generating functions for discrete connected surfaces with one boundary of color 1 and length l (more precisely, surfaces with one marked face of color 1 and of degree l, and one marked edge on the boundary, removed from a closed surface). Similarly, $< \operatorname{Tr} M_2^l >$ is a generating function which counts surfaces with one boundary of color 2 and length l. More generally, $< \operatorname{Tr} M_1^{l_1} \operatorname{Tr} M_1^{l_2} \dots \operatorname{Tr} M_1^{l_m} \operatorname{Tr} M_2^{l'_1} \operatorname{Tr} M_2^{l'_2} \dots \operatorname{Tr} M_2^{l'_{m'}} >_c$ is a generating function which counts connected surfaces with m boundaries of color 1 and respective lengths $l_1, \dots, l_m$, and m' boundaries of color 2 and respective lengths $l'_1, \dots, l'_{m'}$ (see fig.1 for an example). The subscript $<>_c$ in the expectation values means "connected part" or "cumulant", it ensures that only connected surfaces appear in the Wick expansion.

More interesting is:

$$< \operatorname{Tr} M_1^l M_2^{l'} > . \tag{1-3}$$

It is a generating function which counts surfaces with only one boundary of length $l + l'$, with l color 1 sites followed by l' color 2 sites (see fig.2 for an example).

$$\left\langle \mathrm{Tr}\, M_1^2 M_2^5 \right\rangle :=$$

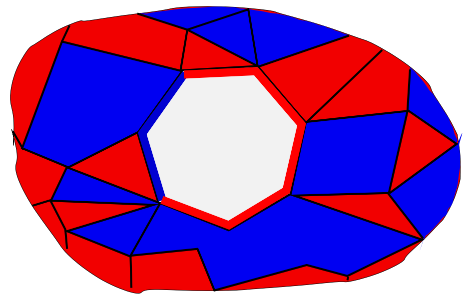


Figure 2: Example of surface generated by $\langle \mathrm{Tr}\, M_1^2 M_2^5 \rangle_c$: it is a disc with one boundary of length 7 with red condition for 5 adjacent segments followed by two segments with blue condition.

And more generally,

$$< \mathrm{Tr}\, M_1^{l_1} M_2^{l'_1} M_1^{l_2} M_2^{l'_2} \dots > \tag{1-4}$$

counts surfaces with one boundary of length $\sum l_i + l'_i$ with l_1 sites of color 1 followed by l'_1 sites of color 2 then l_2 sites of color 1, ..., etc.

It is easy to see that one can design such observables for any given boundary conditions: any number of boundaries, and any pattern of sites on the boundaries.

In this article we show how to compute those generating functions for surfaces of given topology.

1.2 Outline and main results

The paper is organized as follows:

- in section 2, we summarize briefly some previous knowledge of formal 2-matrix model integrals. Namely, we recall how to compute the "disc amplitude", and the spectral curve, and from there the result of [11], i.e. how to count surfaces with uniform boundary conditions.

- in section 3, we define appropriate notations for describing arbitrary boundary conditions. We recall which cases were already known in the literature.

- in section 4, we give the formula for computing the generating functions counting surfaces of any genus and arbitrary boundary conditions. The formula is best represented diagrammatically, and has a very intuitive interpretation.

- in section 5, we show some examples of applications of our formula, and in particular we show how to recover previously known cases.

- section 6 is the conclusion.
- the proof of the main formula of section 4, is written in the appendix, because it is rather technical.

2 Reminder 2-matrix model

The 2-matrix model has generated a considerable number of works. Here, we use the method of loop equations [18, 6, 20], which is well suited for genus expansion computations.

2.1 The resolvent

The resolvent is defined as:

$$\overline{W}_1(x) = \left\langle \mathrm{Tr}\, \frac{1}{x - M_1} \right\rangle = \sum_{l=0}^{\infty} \frac{1}{x^{l+1}} \ \left\langle \mathrm{Tr}\, M_1^l \right\rangle \tag{2-1}$$

it is a generating function for a disc of color 1 (i.e. discrete surface with only one boundary of color 1 and of length l), and x is a complex "fugacity" conjugated to the boundary length l[3].

Like any expectation value in a formal matrix model [3, 6], it admits a topological $1/N^2$ expansion:

$$\overline{W}_1(x) = \sum_{g=0}^{\infty} \overline{W}_1^{(g)}(x) \ N^{1-2g} \tag{2-2}$$

where $\overline{W}_1^{(g)}(x)$ is the generating function for discrete surfaces of genus g.

The loop equations which allow to compute $\overline{W}_1^{(g)}$ have been known for a long time [20]. More recently, $\overline{W}_1^{(g)}$ was computed for any g in [4, 12, 11]. The result for $\overline{W}_1^{(0)}$ can be written in terms of an algebraic equation. Let:

$$y(x) = V_1'(x) - \overline{W}_1^{(0)}(x). \tag{2-3}$$

$y(x)$ is solution of the following algebraic equation [7, 8]:

$$0 = E(x, y(x)) = (V_1'(x) - y(x))(V_2'(y(x)) - x) - P^{(0)}(x, y(x)) + 1 \tag{2-4}$$

where

$$P(x, y) = \left\langle \mathrm{Tr}\, \frac{V_1'(x) - V_1'(M_1)}{x - M_1} \, \frac{V_2'(y) - V_2'(M_2)}{y - M_2} \right\rangle = \sum_{g=0}^{\infty} N^{1-2g} P^{(g)}(x, y) \tag{2-5}$$

and y must be chosen as the branch of the solution of $E(x, y) = 0$ which behaves like $V_1'(x)$ for large x.

[3]Remark that these resolvents are properly defined when the fugacity $x \to \infty$

2.2 The spectral curve

In general, correlation functions are multivalued functions of x, and it is better to write them as functions on a Riemann surface.

Therefore, we view x and y as two meromorphic functions living on a compact Riemann surface Σ.

$$E(x,y)=0 \qquad \leftrightarrow \qquad \exists p\in\Sigma \ / \ x=x(p) \text{ and } y=y(p) \tag{2-6}$$

Since the equation $E(x,y)=0$ has $\deg V_2$ solutions in y for a given x, it means that for every point p in Σ, there are $\deg V_2$ points p^i in Σ such that:

$$\forall i=0,\dots,d_2\ , \qquad x(p^i)=x(p) \tag{2-7}$$

where $d_2=\deg V_2'$, and by convention we assume $p^0=p$.

Similarly, if we regard x as a function of y, then the equation $E(x,y)=0$ has $\deg V_1$ solutions for a given y, which means that for every point p in Σ, there are $\deg V_1$ points $\tilde{p}^i$ in Σ such that:

$$\forall i=0,\dots,d_1\ , \qquad y(\tilde{p}^i)=y(p) \tag{2-8}$$

where $d_1=\deg V_1'$, and by convention we assume $\tilde{p}^0=p$.

2.3 Examples

• If the algebraic curve Σ build from $E(x,y)=0$ has genus zero, it is possible to find a rational parametrization [7, 5], i.e. $x(p)$ and $y(p)$ are rational functions of p:

$$\left\{\begin{array}{l} x(p)=\gamma p+\sum_{k=0}^{\deg V_2'}\alpha_k p^{-k} \\ y(p)=\gamma p^{-1}+\sum_{k=0}^{\deg V_1'}\beta_k p^{k} \end{array}\right. \tag{2-9}$$

where the coefficients α_k, β_k and γ are determined by $y(p)\sim_{p\to\infty} V_1'(x(p))-1/x(p)+O(p^{-2})$ and $x(p)\sim_{p\to 0} V_2'(y(p))-1/y(p)+O(p^2)$.

In that case the compact Riemann surface Σ is the Riemann sphere.

This is the case which counts the Ising model bicolored maps.

• If the algebraic curve Σ build from $E(x,y)=0$ has genus 1, it is possible to find a parametrization with elliptical functions.

Spectral curves $E(x,y)=0$ of genus $g>0$, are not generating functions which counts maps, but they are still solutions of the loop equations, they have a more complicated combinatorical interpretation, and they are very useful for applications to string theory for instance. In what follows, we assume that the spectral curve may have any genus, and one should keep in mind that only the genus zero case really corresponds to the Ising model on random surfaces.

3 Definitions

We assume that the spectral curve $E(x,y)=0$ is known, and that x and y are two meromorphic functions on the compact Riemann surface Σ.

3.1 Notations

The most general boundary condition for a discrete surface generated by the 2-matrix model is made of several boundaries, some of them having color 1, some having color 2, and some having mixed color boundaries.

Let us say that we have:

- m boundaries of color 1, with conjugated parameters $x(p_1),\dots,x(p_m)$,
- n boundaries of color 2, with conjugated parameters $y(q_1),\dots,y(q_n)$,
- l mixed boundaries such that the i^{th} boundary is made of $2k_i$ changes of colors. It can be parameterized with $2k_i$ conjugated length parameters $[x(p_{i,1}),y(q_{i,1}),x(p_{i,2}),y(q_{i,2}),x(p_{i,3}),y(q_{i,3}),\dots,x(p_{i,k}),y(q_{i,k})]$.

Notice that the p_i's and q_j's are points on the curve Σ.

The generating function for discrete surfaces with that boundary condition is:

$$\begin{aligned}
& H_{k_1,\dots,k_l;m;n}(S_1,S_2,\dots,S_l;p_1,\dots,p_m;q_1,\dots,q_n) \\
= \quad & \Big\langle \prod_{i=1}^{l}(N\delta_{k_i,1}+\ \mathrm{Tr}\,\frac{1}{S_i})\ \prod_{j=1}^{m}\mathrm{Tr}\,\frac{1}{x(p_j)-M_1}\prod_{s=1}^{n}\mathrm{Tr}\,\frac{1}{y(q_s)-M_2}\Big\rangle_c \\
& +\delta_{l,0}\delta_{m,2}\delta_{n,0}\frac{1}{(x(p_1)-x(p_2))^2}+\delta_{l,0}\delta_{m,0}\delta_{n,2}\frac{1}{(y(q_1)-y(q_2))^2} \\
& +\delta_{l,0}\delta_{m,1}\delta_{n,0}(y(p_1)-V_1'(x(p_1)))+\delta_{l,0}\delta_{m,0}\delta_{n,1}(x(q_1)-V_2'(y(q_1)))
\end{aligned} \qquad (3-1)$$

where

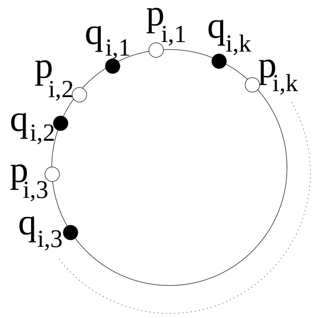
$$\mathrm{Tr}\,\frac{1}{S_i}=\mathrm{Tr}\,\left(\frac{1}{x(p_{i,1})-M_1}\frac{1}{y(q_{i,1})-M_2}\frac{1}{x(p_{i,2})-M_1}\frac{1}{y(q_{i,2})-M_2}\cdots\frac{1}{y(q_{i,k_i})-M_2}\right) \qquad (3\text{-}2)$$

and

$$S_i=[p_{i,1},q_{i,1},p_{i,2},q_{i,2},p_{i,3},q_{i,3},\dots,p_{i,k},q_{i,k}] \qquad (3\text{-}3)$$

is the ordered set of points $\{p_{i,l}, q_{i,l}\}_{l=1\ldots k}$ up to cyclic permutations, i.e., using a graphical representation

$$S_i = \qquad . \tag{3-4}$$



Each p variable stands for a piece of boundary of color 1, whereas each q stands for a piece of color 2.

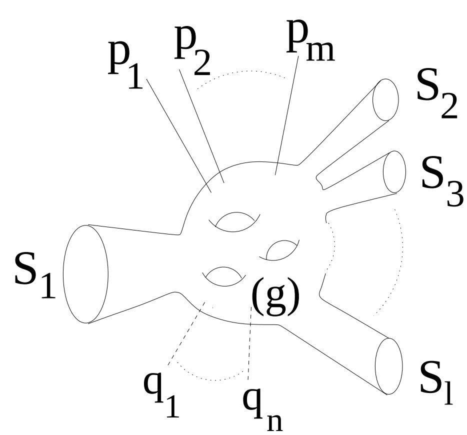
Each $H_{k_1,\ldots,k_l;m;n}$ admits a topological expansion:

$$H_{k_1,\ldots,k_l;m;n} = \sum_{g=0}^{\infty} N^{2-2g-l-m-n} \; H^{(g)}_{k_1,\ldots,k_l;m;n} \tag{3-5}$$

where $H^{(g)}_{k_1,\ldots,k_l;m;n}$ is the generating function for discrete surfaces of genus g with the same boundary conditions (indeed, the Euler characteristic of a surface of genus g with $l+m+n$ boundaries is $\chi = 2-2g-l-m-n$).

We represent $H^{(g)}_{k_1,\ldots,k_l;m;n}$ graphically as a connected surface of genus g, with l circular boundaries, and $n+m$ punctures:

$$H^{(g)}_{\mathbf{k_L};m;n}(\mathbf{S_L}; p_1,\ldots,p_m; q_1,\ldots,q_n) = \qquad . \tag{3-6}$$

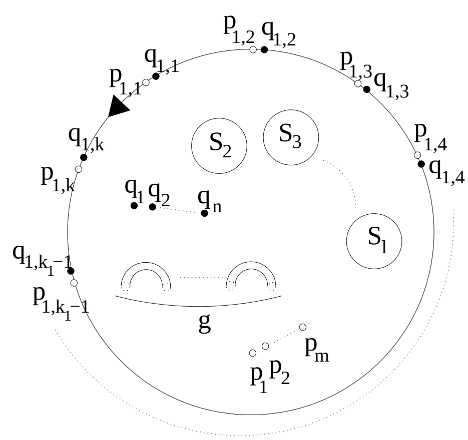


Since the correlation function $H^{(g)}_{k_1,\ldots,k_l;m;n}$ does not depend on the order of the traces (i.e. one may permute the S_i's), we may choose one of the boundaries (for example S_1), and draw it on the exterior, and draw the whole surface in the interior of the circle S_1. Moreover, because of the cyclic invariance of the trace, we may choose a starting point on each boundary (for example $p_{1,1}$) by drawing an anticlockwise arrow on the boundary from this point[4].

[4] Remember that the boundaries are oriented according to the sequence of points in the traces of the correlation functions.

Thus, we represent the correlation function $H^{(g)}_{\mathbf{k_L};m;n}(\mathbf{S_L};p_1,\dots,p_m;q_1,\dots,q_n)$ by a surface $\mathcal{S}^{(g)}_{\mathbf{k_L};m;n}(\mathbf{S_L};p_1,\dots,p_m;q_1,\dots,q_n)$ which is a disc equipped with g handles, $l-1$ holes corresponding to the $l-1$ remaining non homogenous boundaries, m white marked points corresponding to the homogenous boundaries of color 1 and n black marked points corresponding to the homogenous boundaries of color 2. Note also that every non homogenous boundary is equipped with a sequence of white and black points representing the sequence of boundary conditions.

$$H^{(g)}_{\mathbf{k_L};m;n}(\mathbf{S_L};p_1,\dots,p_m;q_1,\dots,q_n) = \tag{3-7}$$



Notice that the other boundaries $S_2,\dots,S_l$ also have a marked edge $p_{i,1}\to q_{i,1}$ whose orientation is opposite (i.e. clockwise) of that of S_1.

3.2 Previously known results

Some cases are already known in the literature:

- **Planar case:** all $H^{(0)}_{k_1,\dots,k_l;0;0}$ (i.e. planar surfaces only) were computed in [14].
- **Non-mixed boundaries:** all functions with only non-mixed boundaries, i.e. $H^{(g)}_{\emptyset;m;n}$ were computed in [12, 4, 11, 15].
- **Only one mixed boundary with $k=1$:** $H^{(g)}_{1;m;n}$ was computed in [15].
- **In particular the sphere with one puncture is the resolvent:**

$$H^{(0)}_{0;1;0}(p) = \overline{W}^{(0)}_1(p) = V_1'(x(p)) - y(p) \tag{3-8}$$

- **In particular the sphere with one bicolored boundary is** [5, 9, 7]:

$$H^{(0)}_{1;0;0}(\{p,q\}) = \frac{E(x(p),y(q))}{(x(p)-x(q))(y(q)-y(p))} \tag{3-9}$$

Below, we compute all the other ones.

4 Diagrammatic solution

Here, we show the recipe to compute recursively any $H_{S_1,\dots,S_l;m;n}$. The proof (which relies on loop equations, and is explained in the appendix is very technical, whereas the solution is rather simple and can be written pictorially.

4.1 In equations

In equations, the recursive solution of the loop equations (see the proof in appendix) can be written:

$$\boxed{\begin{array}{l}
H^{(g)}_{\mathbf{k_L};m;n}(\mathbf{S_L};p_1,\dots,p_m;q_1,\dots,q_n) = \\
\quad \mathop{\mathrm{Res}}_{r\to p_{1,1},p_{i,\alpha},p_j,\tilde{q}^j_{1,k_1}} \frac{H^{(0)}_{1;0;0}(p_{1,1},q_{1,k_1})\,dx(r)}{(x(p_{1,1})-x(r))(y(q_{1,k_1})-y(r))H^{(0)}_{1;0;0}(r,q_{1,k_1})} \times \\
\Big\{ \sum_h \sum_{A\bigcup B=\{2,\dots,l\}} \sum_{\alpha=2}^{k_1} \sum_{I,J} H^{(h)}_{k_1-\alpha+1,\mathbf{k_B};m-|I|;n-|J|}(\{p_{1,\alpha},q_{1,\alpha},\dots p_{1,k_1},q_{1,k_1}\},\mathbf{S_B};\mathbf{p_{M/I}};\mathbf{q_{N/J}}) \\
\qquad \times \frac{H^{(g-h)}_{\alpha-1,\mathbf{k_A};|I|;|J|}(\{r,q_{1,1},\dots p_{1,\alpha-1},q_{1,\alpha-1}\},\mathbf{S_A};\mathbf{p_I};\mathbf{q_J})}{x(p_{1,\alpha})-x(r)} \\
+\sum_{\alpha=2}^{k_1} \frac{1}{x(p_{1,\alpha})-x(r)} \times \\
H^{(g-1)}_{\alpha-1,k_1-\alpha+1,\mathbf{k_{L/\{1\}}};m;n}(\{r,q_{1,1},\dots p_{1,\alpha-1},q_{1,\alpha-1}\},\{p_{1,\alpha},q_{1,\alpha},\dots p_{1,k_1},q_{1,k_1}\},\mathbf{S_{L/\{1\}}};\mathbf{p_M};\mathbf{q_N}) \\
+\sum_{i=2}^{l}\sum_{\alpha=1}^{k_i} \frac{1}{x(p_{i,\alpha})-x(r)} \times \\
\quad \times H^{(g)}_{k_1+k_i,\mathbf{k_{L/\{1,i\}}};m;n}(\{S_1(r),p_{i,\alpha},q_{i,\alpha},p_{i,\alpha+1},\dots,q_{i,k_i},p_{i,1},\dots,p_{i,\alpha-1},q_{i,\alpha-1}\},\mathbf{S_{L/\{1,i\}}};\mathbf{p_M};\mathbf{q_N}) \\
+\sum_h \sum_{A\bigcup B=\{2,\dots,l\}} \sum_{I,J} H^{(h)}_{k_1,\mathbf{k_A};|I|;|J|}(S_1(r),\mathbf{S_A};\mathbf{p_I};\mathbf{q_J}) H^{(g-h)}_{\mathbf{k_B};m-|I|+1;n-|J|}(\mathbf{S_B};r,\mathbf{p_{M/I}};\mathbf{q_N}/\mathbf{J}) \\
+\sum_{h=1}^{g} H^{(h)}_{0;1;0}(r) H^{(g-h)}_{k_1,\dots,k_l;m;n}(S_1(r),S_2,\dots,S_l;p_1,\dots,p_m;q_1,\dots,q_n) \\
+H^{(g-1)}_{\mathbf{k_L};m+1;n}(\mathbf{S_K}(r);r,\mathbf{p_M};\mathbf{q_N})\Big\}
\end{array}}$$

(4-1)

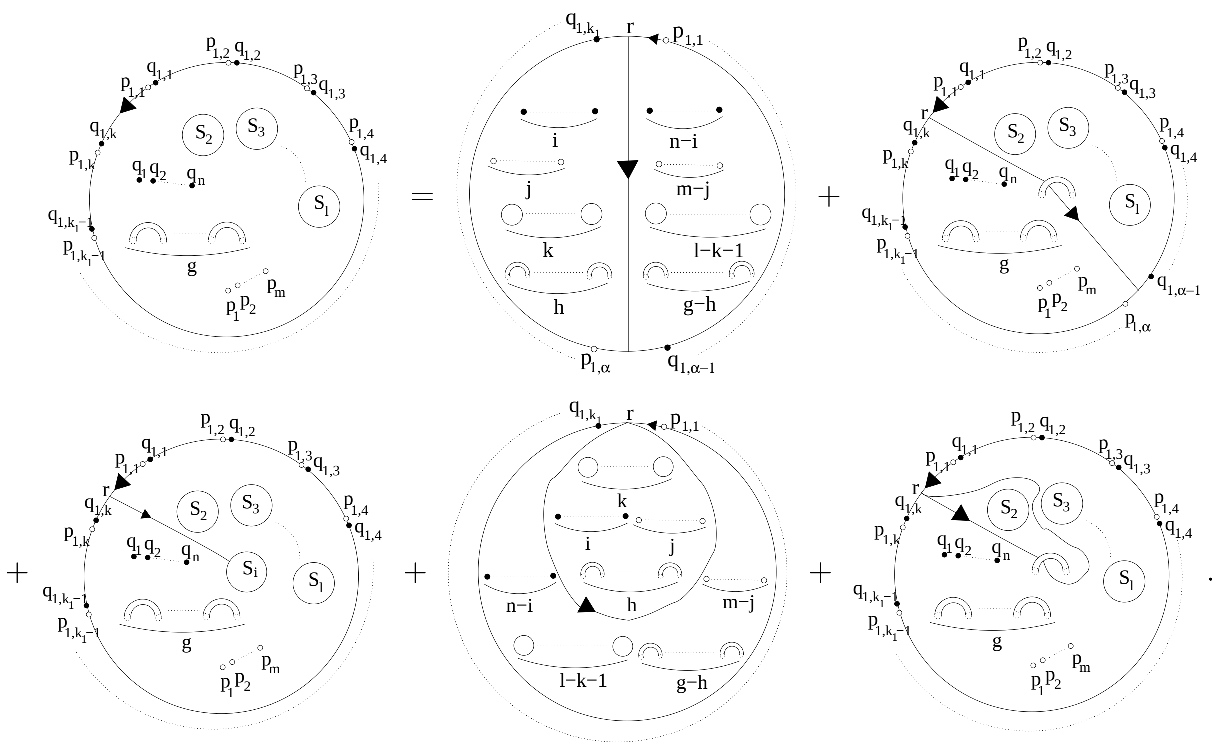
It looks terrible, but each term can be represented diagrammatically, and it is in fact rather simple and intuitive. Let us notice for the moment that this formula involves residues (i.e. contour integrals on Σ) at various points, in particular the $\tilde{q}^j_{1,k_1}$ which are defined in eq.2-8, and were we mean $j \neq 0$.

This formula also involves the function $H^{(0)}_{1;0;0}$ which is given in eq.3-9.

All the other terms in the RHS of eq.4-1 are either some $H^{(g)}_{S;m;n}$'s computed recursively by the same formula, or some $H^{(g)}_{0;m;n}$ which were computed in [12, 4, 11, 15].

4.2 Diagrammatic representation

It is more convenient to represent equation 4-1 diagrammatically:



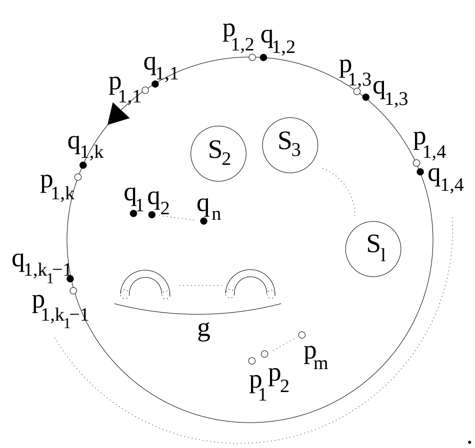
$$(4\text{-}2)$$

where we explain the meaning of those graphs below.

4.2.1 Cutting surfaces

Consider a connected surface $\mathcal{S}$ with at least one boundary (i.e. $l \geq 1$):

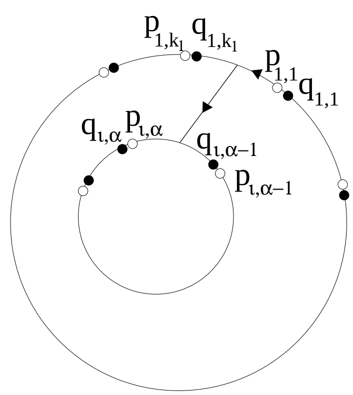
$$\mathcal{S} = \mathcal{S}^{(g)}_{\mathbf{k_L};m;n}(\mathbf{S_L}; p_1, \ldots, p_m; q_1, \ldots, q_n) = \tag{4-3}$$



Let Cut($\mathcal{S}$) be the set of all topologically inequivalent possibilities of cutting the surface along a line $p_{1,1} \to p_{i,\alpha}$ (we allow the closed line $(i,\alpha) = (1,1)$). When we cut along such a line, we can either get a connected or a disconnected surface. The only possibility of getting a disconnected surface is if the point $p_{i,\alpha}$ belongs to S_1, i.e. $i = 1$, and if there is no handle going above the cut.

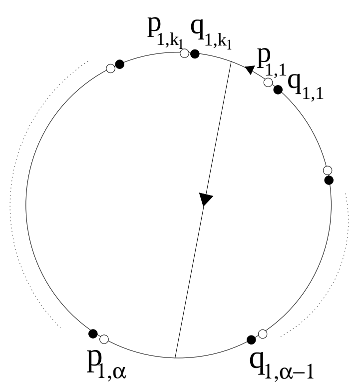
Here is the algorithm to construct $\mathrm{Cut}(\mathcal{S})$:

- one first has to choose any ending point $p_{i,\alpha}$ on a mixed boundary and draw a path going from the left of the starting point to the left of the ending point[5]:
  - This point can belong to a boundary different from the starting one, $i \neq 1$:



(4-4)

    There are $\sum_{i=2}^{l} k_i$ such possibilities.
  - It can belong to the same boundary, $i = 1, \alpha \neq 1$:

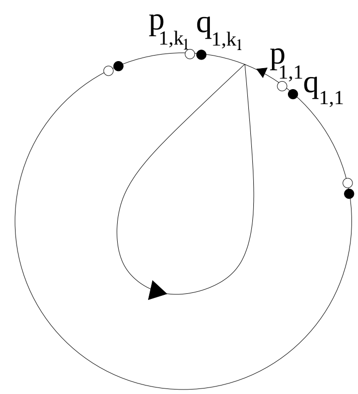


(4-5)

    There are $k_1 - 1$ such possibilities.

[5]The orientation is seen from the point of view of an observer living on the upper side of the disc.

– It can be the same as the starting point $i = 1, \alpha = 1$:

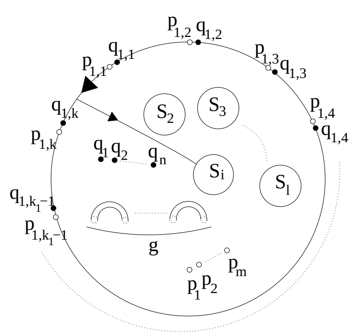


(4-6)

There is only one such possibility.

- Once this ending point is chosen, it remains to fix the position of the handles and the other boundaries and punctures with respect to this path. The number of inequivalent possibilities depends on the respective position of the starting and ending points:

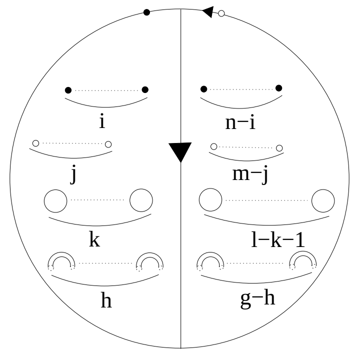
  – If the starting and ending points do not belong to the same boundary, the surface is not disconnected by the cut, and every choices are equivalent since the left and right side of the path belong to the same component of the surface:



(4-7)

  There is only one possibility for the boundaries, punctures and handles configuration.

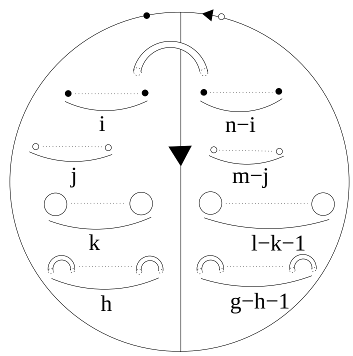
  – If the starting and ending points belong to the same boundary, two different configurations can occur: either the path does cut the disc into two disconnected parts, i.e. no handle goes above the path. In this case, one has to choose for each handle and boundary whether it lies to the left or the right of the path:



(4-8)

There are 2^{n+m+g} such configurations;

Either the path does not separate the disc into two parts, and all the positions of handles, punctures and boundaries are equivalent:



(4-9)

There is only one such configuration because one can transport the handles, punctures and boundaries across the handle above the path.

We have then built the set $\mathrm{Cut}(\mathcal{S})$ of cut surfaces associated to any surface $\mathcal{S}$.

4.3 Weights of graphs

Now, let us associate a weight to each cut surface. We define recursively a weight $\mathcal{P}$ on the set of graphs:

Definition 4.1 *The weight $\mathcal{P}$ of an uncut surface is given by the corresponding correlation function:*

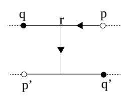
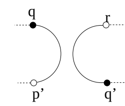
$$\mathcal{P}\left(\mathcal{S}^{(g)}_{\mathbf{k_L};m;n}(\mathbf{S_L};p_1,\dots,p_m;q_1,\dots,q_n)\right) := H^{(g)}_{\mathbf{k_L};m;n}(\mathbf{S_L};p_1,\dots,p_m;q_1,\dots,q_n). \quad (4\text{-}10)$$

The weight of the disconnected union of two surfaces is the product of their respective weights:

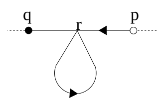
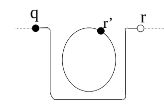
$$\mathcal{P}(\mathcal{S}\bigcup\mathcal{S}') := \mathcal{P}(\mathcal{S})\times\mathcal{P}(\mathcal{S}'). \quad (4\text{-}11)$$

The weight of a cut surface is obtained from the weight of the surface(s) obtained by cutting along the path γ following the rules:

- *If the starting and ending points of γ do not coincide:*

$\mathcal{P}\Big($  $\Big)$ $= \mathop{\mathrm{Res}}_{r\to p,\tilde{q}^j,p'} \dfrac{H^{(0)}_{1;0;0}(p,q)}{(x(p)-x(r))(y(q)-y(r))(x(p')-x(r))H^{(0)}_{1;0;0}(r,q)}\mathcal{P}\Big($  $\Big)$ (4-12)

- *If the starting and ending points coincide:*

$\mathcal{P}\Big($  $\Big)$ $= \mathop{\mathrm{Res}}_{r\to p,\tilde{q}^j,p_i} \dfrac{H^{(0)}_{1;0;0}(p,q)}{(x(p)-x(r))(y(q)-y(r))H^{(0)}_{1;0;0}(r,q)}\mathcal{P}\Big($  $\Big)$ (4-13)

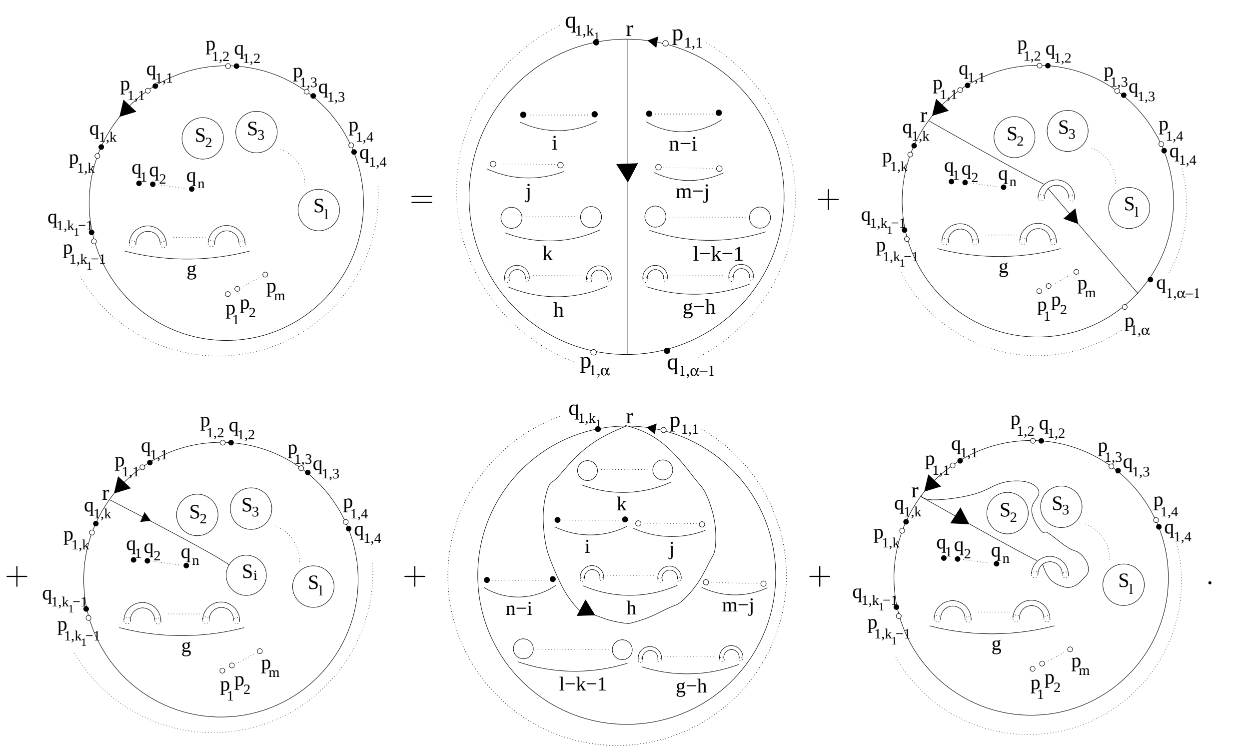
where the p_i's are the points encircled inside the closed loop.

With such notations, equation 4-1 can be reinterpreted as:

Theorem 4.1 *The weight of a given surface is equal to the sum of the weights of all corresponding cut surfaces:*

$$\mathcal{P}(\mathcal{S}) = \sum_{S\in\mathrm{cut}(\mathcal{S})} \mathcal{P}(S) \tag{4-14}$$

I.e. graphically:



(4-15)

Performing this procedure recursively on any correlation functions, one can eliminate the mixed boundaries step by step until there is no mixed boundary left, i.e.

until there are only punctures left. The correlation functions with only punctures are computed in [12, 4, 11].

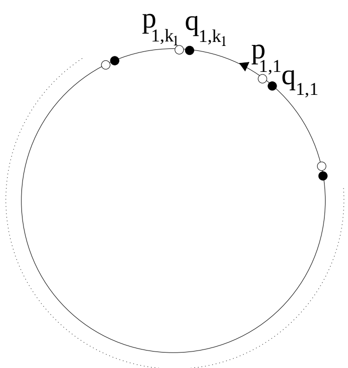
5 Examples of applications

In this section, we show how to use our formula to recover some previously known results, in particular the planar case, and surfaces with uniform boundaries. We also compute two simple examples: the generating function of discs with four boundary operators and the generating function of cylinders with two boundary operators on each boundary.

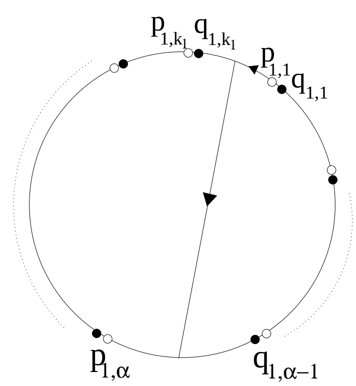
5.1 Link with former results

5.1.1 Planar mixed traces

If one is interested in the planar mixed correlation functions with only one boundary, the recursion relation simplifies to:



$$= \sum_{\alpha=2}^{k_1}$$



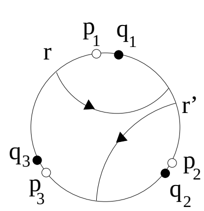
(5-1)

One can thus draw the result of the whole recursive procedure as the sum over all possible link patterns on the starting disc in such a way they separate all boundary variables. This reproduce the decomposition used in [14] to compute the building blocks $F_k = C^k_{id}$.

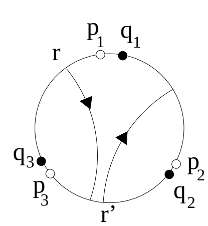
Example:

The three point mixed correlation function reads:

$$H^{(0)}_{3;0;0}(p_1, q_1, p_2, q_2, p_3, q_3) =$$



$+$



(5-2)

which gives:

$$H^{(0)}_{3;0;0}(p_1,q_2,p_2,q_2,p_3,q_3) =$$
$$\mathop{\mathrm{Res}}_{r\to p_1,p_2,\tilde q_3^j}\mathop{\mathrm{Res}}_{r'\to p_2,p_3,\tilde q_3^j} \frac{H^{(0)}_{1;0;0}(p_1,q_3)H^{(0)}_{1;0;0}(r,q_1)H^{(0)}_{1;0;0}(p_2,q_3)H^{(0)}_{1;0;0}(p_3,q_3)H^{(0)}_{1;0;0}(r',q_2)}{(x(p_1)-x(r))(y(q_3)-y(r))(x(p_2)-x(r))H^{(0)}_{1;0;0}(r,q_3)(x(p_2)-x(r'))(y(q_3)-y(r'))(x(p_3)-x(r'))H^{(0)}_{1;0;0}(r',q_3)}$$
$$+\mathop{\mathrm{Res}}_{r\to p_1,p_3,\tilde q_3^j}\mathop{\mathrm{Res}}_{r'\to r,p_2,\tilde q_2^j} \frac{H^{(0)}_{1;0;0}(p_1,q_3)H^{(0)}_{1;0;0}(r,q_2)H^{(0)}_{1;0;0}(p_3,q_3)H^{(0)}_{1;0;0}(p_2,q_2)H^{(0)}_{1;0;0}(r',q_1)}{(x(p_1)-x(r))(y(q_3)-y(r))(x(p_3)-x(r))H^{(0)}_{1;0;0}(r,q_3)(x(r)-x(r'))(y(q_2)-y(r'))(x(p_2)-x(r'))H^{(0)}_{1;0;0}(r',q_2)} \tag{5-3}$$

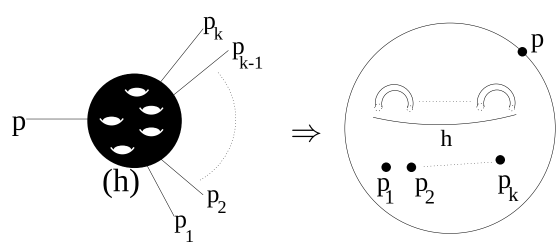
One can easily show that this coincide with the result of [14] by using explicitly the orientation-reversing symmetry[6] of the correlation function:

$$H^{(0)}_{3;0;0}(p_1,q_1,p_2,q_2,p_3,q_3) = H^{(0)}_{3;0;0}(p_1,q_3,p_3,q_2,p_2,q_1). \tag{5-4}$$

5.1.2 Simple traces topological expansion

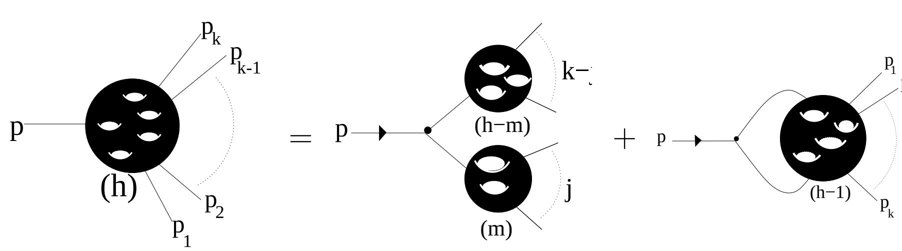
One can remark that all this recursive procedure supposes that the non-mixed correlation functions are known, since this new diagrammatic representation does not allow to compute them. Nevertheless they were computed by a similar procedure in [12, 4, 11] in terms of trivalent graphs and we show that these former rules could be written in a graphical representation similar to the one presented in this paper.

Let us represent $W^{(h)}_{k+1}(p,p_1,\dots,p_k) := H^{(h)}_{0,k+1,0}(p,p_1,\dots,p_k)$ as a disk with k punctures instead of a sphere with $k+1$ punctures (we have drawn the surface generated by this function inside the boundary corresponding to p):



(5-5)

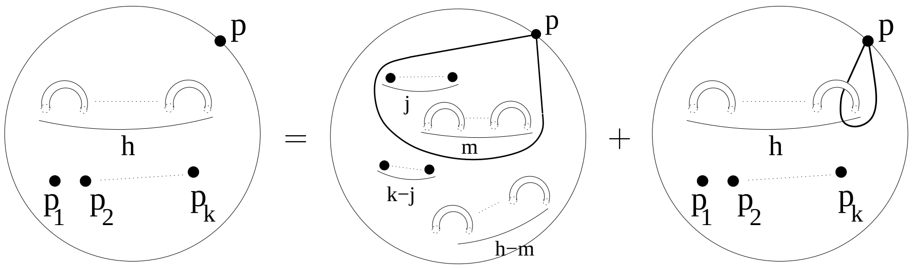
The recursion relation of [4, 11]



(5-6)

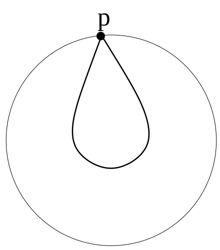
[6]Combinatoricaly, this means that summing over all oriented surfaces is equivalent to summing over all surfaces with the orientation reversed.

can then be written:

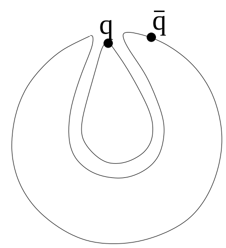


(5-7)

where once again, the different terms on the RHS are obtained by drawing a basis of homologically independent paths on the disk starting and ending on the boundary, and the weight of a cutting along this path follows:



$$= \sum_i \mathop{\mathrm{Res}}_{q \to a_i} \frac{dE_q(p)}{(y(q) - y(\overline{q}))dx(q)} \tag{5-8}$$

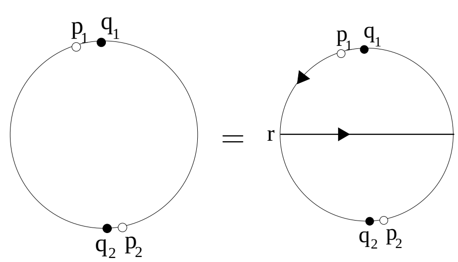


where one sums over all branch points a_i and $\overline{q}$ is the point conjugated to q (see [4, 11] for more details).

5.2 Four point function on the disc

The correlation function $H^{(0)}_{2;0;0}(p_1, q_1, p_2, q_2)$ has already been computed in [8, 14]. Nevertheless, this computation used an Ansatz and a symmetry property of the correlation function explicitly. Let us recover the same result without using any symmetry consideration, but using our recursive formula instead.

The solution of the loop equations reads graphically:



(5-9)

which is translated into[7]

$$H_2^{(0)}(p_1, q_1, p_2, q_2) = \mathop{\mathrm{Res}}_{r \to p_1, p_2, \tilde{q}_2^j} \frac{H_1^{(0)}(p_1, q_2) H_1^{(0)}(p_2, q_2)\, dx(r)}{H_1^{(0)}(r, q_2)(x(p_1) - x(r))(y(q_2) - y(r))} \frac{H_1^{(0)}(r, q_1)}{x(p_2) - x(r)}. \tag{5-10}$$

[7]For shortening the notations, we write all along this section $H_k^{(g)}(p_1, q_1, p_2, q_2, \ldots, p_k, q_k) := H^{(g)}_{k;0;0}(p_1, q_1, p_2, q_2, \ldots, p_k, q_k)$.

Writing

$$\frac{1}{y(q_2)-y(r)} = \frac{y(r)-y(q_1)}{(y(q_2)-y(q_1))(y(q_2)-y(r))} + \frac{1}{y(q_2)-y(q_1)} \tag{5-11}$$

one gets

$$\begin{aligned} & H_2^{(0)}(p_1,q_1,p_2,q_2) \\ = & \operatorname*{Res}_{r\to p_1,p_2,\tilde{q}_2^j} \frac{H_1^{(0)}(p_1,q_2)H_1^{(0)}(p_2,q_2)\,dx(r)}{H_1^{(0)}(r,q_2)(x(p_1)-x(r))(y(q_2)-y(q_1))} \frac{H_1^{(0)}(r,q_1)}{x(p_2)-x(r)} \\ + & \operatorname*{Res}_{r\to p_1,p_2,\tilde{q}_2^j} \frac{H_1^{(0)}(p_1,q_2)H_1^{(0)}(p_2,q_2)\,dx(r)}{(x(p_1)-x(r))(x(p_2)-x(r))(y(q_2)-y(q_1))} \frac{(y(r)-y(q_1))H_1^{(0)}(r,q_1)}{(y(q_2)-y(r))H_1^{(0)}(r,q_2)}. \end{aligned} \tag{5-12}$$

Since

$$H_1^{(0)}(p,q)(y(q)-y(p)) = \frac{E(x(p),y(q))}{x(p)-x(q)} \tag{5-13}$$

the integrand of the second term in the RHS is a rational function in $x(r)$ and it is easily checked that the integration contour encircles all its poles (this function is regular when $x(r)\to\infty$). Thus this second term vanishes.

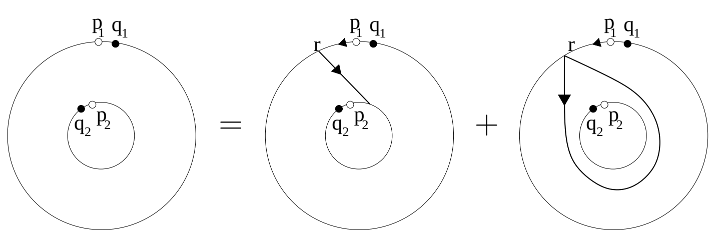
The first term has no pole at $r=\tilde{q}_2^j$, thus it involves only simple poles when $r\to p_1,p_2$ and we recover the known result [8, 14]:

$$H_2^{(0)}(p_1,q_1,p_2,q_2) = -\frac{H_1^{(0)}(p_1,q_1)H_1^{(0)}(p_2,q_2) - H_1^{(0)}(p_1,q_1)H_1^{(0)}(p_2,q_2)}{(x(p_1)-x(p_2))(y(q_1)-y(q_2))}. \tag{5-14}$$

Even if this new derivation of an old result seems more involved technically, it has the advantage of being constructive (the derivation of [14] was based on an ansatz) and does not suppose any additional symmetry of the correlation functions (the derivation of [8] was based on the fact that $H_2^{(0)}(p_1,q_1,p_2,q_2) = H_2^{(0)}(p_1,q_2,p_2,q_1)$).

5.3 Generating function of cylinders

The generating function of cylinders with 2 boundary operators on both boundaries is obtained by:



(5-15)

which can be translated into

$$H_{1,1;0;0}^{(0)}(\{p_1,q_1\},\{p_2,q_2\}) \quad = \quad \operatorname*{Res}_{r\to p_1,p_2,\tilde{q}_1^j} \frac{H_1^{(0)}(p_1,q_1)\,dx(r)}{H_1^{(0)}(r,q_1)(x(p_1)-x(r))(y(q_1)-y(r))} \times$$

$$\times \left[\frac{H_2^{(0)}(r, q_1, p_2, q_2)}{x(p_2) - x(r)} + H_1^{(0)}(r, q_1) H_{1;1;0}^{(0)}(p_2, q_2; r) \right] \qquad (5-16)$$

where the second term was computed in [15]:

$$H_{1;1;0}^{(0)}(p_2, q_2; r) = \operatorname*{Res}_{r' \to p_2, r, \tilde{q}_2^j} \frac{H_1^{(0)}(p_2, q_2) H_{0;2;0}^{(0)}(r; r')\, dx(r)}{(x(p_2) - x(r'))(y(r') - y(q_2))}. \qquad (5\text{-}17)$$

and $H_{0;2;0}^{(0)}(r; r')\, dx(r) dx(r')$ is the Bergmann kernel.

6 Conclusion

In this article we have found a recursive and graphical method to compute correlation functions corresponding to every possible boundary condition for the 2-matrix model, i.e. bicolored discrete surfaces.

The result seems to have a nice combinatorial interpretation, as all the possibilities of drawing interfaces (between the + and - spins of the Ising model) in all possible ways. However, a combinatorial derivation is missing.

Also, our result can have interpretations in conformal field theories when one goes to the so called double-scaling-limit [6, 5], and should be compared with recent results from Liouville theory [19, 16, 2]. In particular, in [2], our formula for planar disc amplitudes is interpreted in terms of the interactions of long folded strings and it would be interesting to check the non-planar cases as well.

It would be interesting also to understand how the structure we exhibit in this article, and which seems to be related to integrability like in [14], is related to the Langlands programm as claimed by [16].

Acknowledgments

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Appendix A Loop equations

Here we prove the equation 4-1, using loop equations. Loop equations is a standard and powerful tool in random matrix theory, they are the Ward identities, or Schwinger-Dyson equations, they implement the Virasoro or W-algebra constraints, in combinatorics they can be viewed as an extension of Tutte's equations, and in fact they just consist in integration by parts, or said differentely, the fact that an integral is invariant under (an infinitesimal) change of variable.

For the 2-matrix model, loop equations were first exploited by Staudacher [20], and then by many authors, and they led to the solution of [12, 4, 15].

A.1 The loop equations

In order to prove eq.4-1, we consider the change of variables

$$\begin{aligned}\delta M_1 \quad := \quad & \frac{1}{x(p_{1,1})-M_1}\frac{1}{y(p_{1,1})-M_2}\frac{1}{x(p_{1,2})-M_1}\frac{1}{y(p_{1,2})-M_2}\cdots\frac{1}{x(p_{1,k_1})-M_1}\frac{1}{y(p_{1,k_1})-M_2} \\ & \prod_{i=2}^{l} \mathrm{Tr}\,\left(\frac{1}{x(p_{i,1})-M_1}\frac{1}{y(p_{i,1})-M_2}\frac{1}{x(p_{i,2})-M_1}\frac{1}{y(p_{i,2})-M_2}\cdots\frac{1}{x(p_{i,k_i})-M_1}\frac{1}{y(p_{i,k_i})-M_2}\right) \\ & \quad \prod_{j=1}^{m} \mathrm{Tr}\,\frac{1}{x(p_j)-M_1}\prod_{s=1}^{n} \mathrm{Tr}\,\frac{1}{y(q_s)-M_2}.\end{aligned} \tag{1-1}$$

Writing that the matrix integral is invariant under this change of variable gives the loop equation:

$$\begin{aligned}&(Y(p_{1,1})-y(q_{1,k_1})-Pol_{x(p_{1,1})}V_1'(x(p_{1,1})))H_{k_1,\ldots,k_l;m;n}(S_1,S_2,\ldots,S_l;p_1,\ldots,p_m;q_1,\ldots,q_n) = \\ &\sum_{A\bigcup B=\{2,\ldots,l\}}\sum_{I,J} H_{k_1,\mathbf{k_A};|I|;|J|}(S_1,\mathbf{S_A};\mathbf{p_I};\mathbf{q_J})H_{\mathbf{k_B};m-|I|+1;n-|J|}(\mathbf{S_B};p_{1,1}\mathbf{p_{M/I}};\mathbf{q_N}/\mathbf{J}) \\ &+\sum_{A\bigcup B=\{2,\ldots,l\}}\sum_{\alpha=2}^{k_1}\sum_{I,J} H_{k_1-\alpha+1,\mathbf{k_B};m-|I|;n-|J|}(\{p_{1,\alpha},q_{1,\alpha},\ldots p_{1,k_1},q_{1,k_1}\},\mathbf{S_B};\mathbf{p_{M/I}};\mathbf{q_{N/J}}) \\ &\qquad\times\frac{H_{\alpha-1,\mathbf{k_A};|I|;|J|}(\{p_{1,1},q_{1,1},\ldots p_{1,\alpha-1},q_{1,\alpha-1}\},\mathbf{S_A};\mathbf{p_I};\mathbf{q_J})-H_{\alpha-1,\mathbf{k_A};|I|;|J|}(\{p_{1,\alpha},q_{1,1},\ldots p_{1,\alpha-1},q_{1,\alpha-1}\},\mathbf{S_A};\mathbf{p_I};\mathbf{q_J})}{x(p_{1,\alpha})-x(p_{1,1})} \\ &\sum_{i=2}^{l}\sum_{\alpha=1}^{k_i}\frac{1}{x(p_{i,\alpha})-x(p_{1,1})} \\ &\Big[H_{k_1+k_i,\mathbf{k_{L/\{1,i\}}};m;n}(\{S_1,p_{i,\alpha},q_{i,\alpha},p_{i,\alpha+1},\ldots,q_{i,k_i},p_{i,1},\ldots,p_{i,\alpha-1},q_{i,\alpha-1}\},\mathbf{S_{L/\{1,i\}}};\mathbf{p_M};\mathbf{q_N}) \\ &-H_{k_1+k_i,\mathbf{k_{L/\{1,i\}}};m;n}(\{S_1,p_{i,\alpha},q_{i,\alpha},p_{i,\alpha+1},\ldots,q_{i,k_i},p_{i,1},\ldots,p_{i,\alpha-1},q_{i,\alpha-1}\},\mathbf{S_{L/\{1,i\}}};\mathbf{p_M};\mathbf{q_N})\Big|_{p_{1,1}:=p_{i,\alpha}}\Big] \\ &-\sum_{i=1}^{m}\partial_{p_i}\left[\frac{H_{\mathbf{k_L};m-1;n}(\mathbf{S_L};\mathbf{p_{M/\{i\}}};\mathbf{q_N})-H_{\mathbf{k_L};m-1;n}(\mathbf{S_L};\mathbf{p_{M/\{i\}}};\mathbf{q_N})\big|_{p_{1,1}:=p_i}}{x(p_i)-x(p_{1,1})}\right] \\ &+\frac{1}{N^2}\sum_{\alpha=2}^{k_1}\frac{1}{x(p_{1,\alpha})-x(p_{1,1})}\times \\ &\Big[H_{\alpha-1,k_1-\alpha+1,\mathbf{k_{L/\{1\}}};m;n}(\{p_{1,1},q_{1,1},\ldots p_{1,\alpha-1},q_{1,\alpha-1}\},\{p_{1,\alpha},q_{1,\alpha},\ldots p_{1,k_1},q_{1,k_1}\},\mathbf{S_{L/\{1\}}};\mathbf{p_M};\mathbf{q_N}) \\ &-H_{\alpha-1,k_1-\alpha+1,\mathbf{k_{L/\{1\}}};m;n}(\{p_{1,\alpha},q_{1,1},\ldots p_{1,\alpha-1},q_{1,\alpha-1}\},\{p_{1,\alpha},q_{1,\alpha},\ldots p_{1,k_1},q_{1,k_1}\},\mathbf{S_{L/\{1\}}};\mathbf{p_M};\mathbf{q_N})\Big] \\ &+\frac{1}{N^2}H_{\mathbf{k_L};m+1;n}(\mathbf{S_K};p_{1,1},\mathbf{p_M};\mathbf{q_N})\end{aligned} \tag{1-2}$$

where $Pol_x f(x)$ denotes the polynomial part in x of f, i.e. the sum of the positive terms in the large x Laurent expansion of $f(x)$.

Let us now write its g^{th} order in the topological expansion:

$$
\begin{aligned}
&(y(p_{1,1}) - y(q_{1,k_1}) - Pol_{x(p_{1,1})} V_1'(x(p_{1,1}))) H^{(g)}_{k_1,\dots,k_l;m;n}(S_1, S_2, \dots, S_l; p_1, \dots, p_m; q_1, \dots, q_n) = \\
&\sum_{h=1}^{g} H^{(h)}_{0;1;0}(p_{1,1}) H^{(g-h)}_{k_1,\dots,k_l;m;n}(S_1, S_2, \dots, S_l; p_1, \dots, p_m; q_1, \dots, q_n) \\
&+ \sum_h \sum_{A \bigcup B = \{2,\dots,l\}} \sum_{I,J} H^{(h)}_{k_1,\mathbf{k_A};|I|;|J|}(S_1, \mathbf{S_A}; \mathbf{p_I}; \mathbf{q_J}) H^{(g-h)}_{\mathbf{k_B};m-|I|+1;n-|J|}(\mathbf{S_B}; p_{1,1}\mathbf{p_{M/I}}; \mathbf{q_N/J}) \\
&+ \sum_h \sum_{A \bigcup B = \{2,\dots,l\}} \sum_{\alpha=2}^{k_1} \sum_{I,J} H^{(h)}_{k_1-\alpha+1,\mathbf{k_B};m-|I|;n-|J|}(\{p_{1,\alpha}, q_{1,\alpha}, \dots p_{1,k_1}, q_{1,k_1}\}, \mathbf{S_B}; \mathbf{p_{M/I}}; \mathbf{q_{N/J}}) \\
&\quad \times \frac{H^{(g-h)}_{\alpha-1,\mathbf{k_A};|I|;|J|}(\{p_{1,1},q_{1,1},\dots p_{1,\alpha-1},q_{1,\alpha-1}\},\mathbf{S_A};\mathbf{p_I};\mathbf{q_J}) - H^{(g-h)}_{\alpha-1,\mathbf{k_A};|I|;|J|}(\{p_{1,\alpha},q_{1,1},\dots p_{1,\alpha-1},q_{1,\alpha-1}\},\mathbf{S_A};\mathbf{p_I};\mathbf{q_J})}{x(p_{1,\alpha})-x(p_{1,1})} \\
&+ \sum_{i=2}^{l} \sum_{\alpha=1}^{k_i} \frac{1}{x(p_{i,\alpha})-x(p_{1,1})} \\
&\Big[H^{(g)}_{k_1+k_i,\mathbf{k_{L/\{1,i\}}};m;n}(\{S_1, p_{i,\alpha}, q_{i,\alpha}, p_{i,\alpha+1}, \dots, q_{i,k_i}, p_{i,1}, \dots, p_{i,\alpha-1}, q_{i,\alpha-1}\}, \mathbf{S_{L/\{1,i\}}}; \mathbf{p_M}; \mathbf{q_N}) \\
&- H^{(g)}_{k_1+k_i,\mathbf{k_{L/\{1,i\}}};m;n}(\{S_1, p_{i,\alpha}, q_{i,\alpha}, p_{i,\alpha+1}, \dots, q_{i,k_i}, p_{i,1}, \dots, p_{i,\alpha-1}, q_{i,\alpha-1}\}, \mathbf{S_{L/\{1,i\}}}; \mathbf{p_M}; \mathbf{q_N}) \Big|_{p_{1,1}:=p_{i,\alpha}} \Big] \\
&+ \sum_{i=1}^{m} \partial_{p_i} \left[\frac{H_{\mathbf{k_L};m-1;n}(\mathbf{S_L}; \mathbf{p_{M/\{i\}}}; \mathbf{q_N})\big|_{p_{1,1}:=p_i}}{x(p_i)-x(p_{1,1})} \right] \\
&+ \sum_{\alpha=2}^{k_1} \frac{1}{x(p_{1,\alpha})-x(p_{1,1})} \times \\
&\Big[H^{(g-1)}_{\alpha-1,k_1-\alpha+1,\mathbf{k_{L/\{1\}}};m;n}(\{p_{1,1}, q_{1,1}, \dots p_{1,\alpha-1}, q_{1,\alpha-1}\}, \{p_{1,\alpha}, q_{1,\alpha}, \dots p_{1,k_1}, q_{1,k_1}\}, \mathbf{S_{L/\{1\}}}; \mathbf{p_M}; \mathbf{q_N}) \\
&- H^{(g-1)}_{\alpha-1,k_1-\alpha+1,\mathbf{k_{L/\{1\}}};m;n}(\{p_{1,\alpha}, q_{1,1}, \dots p_{1,\alpha-1}, q_{1,\alpha-1}\}, \{p_{1,\alpha}, q_{1,\alpha}, \dots p_{1,k_1}, q_{1,k_1}\}, \mathbf{S_{L/\{1\}}}; \mathbf{p_M}; \mathbf{q_N}) \Big] \\
&+ H^{(g-1)}_{\mathbf{k_L};m+1;n}(\mathbf{S_K}; p_{1,1}, \mathbf{p_M}; \mathbf{q_N}).
\end{aligned}
\tag{1-3}
$$

Notice that we have used the normalizations $H_{0;2;0} = \langle \mathrm{Tr}\ \mathrm{Tr} \rangle_c + \frac{1}{(x-x)^2}$ explicitly.

A.2 Solution of the equations

We can solve this hierarchy of equations by induction in the number of traces in the correlations and the genus. Indeed, one can remark that the RHS of Eq. (1-3) contains correlation functions with either less traces (that is to say less arguments) either lower genus compare to the correlation function in the LHS. One also knows that the last term of the LHS is a polynomial in $x(p_{1,1})$ of degree $d_1 - 1$ and one can compute its value in the d_1 points $p_{1,1} \to \tilde{q}^j_{1,k_1}$ for $j = 1 \dots d_1$ independently of $H^{(g)}_{\mathbf{k_L};m;n}(\mathbf{S_K}; \mathbf{p_M}; \mathbf{q_N})$.

For this purpose, let us study the behavior of the LHS when $p_{1,1} \to \tilde{q}^j_{1,k_1}$. If p lies in the x-physical sheet and q_{1,k_1} to the y-physical sheet, the definition of the correlation function reads:

$$
\begin{aligned}
&(y(p_{1,1}) - y(q_{1,k_1})) H^{(g)}_{k_1,\dots,k_l;m;n}(S_1, S_2, \dots, S_l; p_1, \dots, p_m; q_1, \dots, q_n) = \\
&= (y(p_{1,1}) - y(q_{1,k_1})) \Big\langle \prod_{i=1}^{l} (N\delta_{k_i,1} + \mathrm{Tr}\ \frac{1}{S_i}) \prod_{j=1}^{m} \mathrm{Tr}\ \frac{1}{x(p_j) - M_1} \prod_{s=1}^{n} \mathrm{Tr}\ \frac{1}{y(q_s) - M_2} \Big\rangle^{(g)}_c \\
&= -\Big\langle (N\delta_{k_i,1} + \mathrm{Tr}\ \frac{1}{\widehat{S}_1}) \prod_{i=2}^{l} (N\delta_{k_i,1} + \mathrm{Tr}\ \frac{1}{S_i}) \prod_{j=1}^{m} \mathrm{Tr}\ \frac{1}{x(p_j) - M_1} \prod_{s=1}^{n} \mathrm{Tr}\ \frac{1}{y(q_s) - M_2} \Big\rangle^{(g)}_c
\end{aligned}
$$

$$+\left\langle (N\delta_{k_i,1} + \mathrm{Tr}\, \frac{1}{\check{S}_1}) \prod_{i=2}^{l} (N\delta_{k_i,1} + \mathrm{Tr}\, \frac{1}{S_i}) \prod_{j=1}^{m} \mathrm{Tr}\, \frac{1}{x(p_j) - M_1} \prod_{s=1}^{n} \mathrm{Tr}\, \frac{1}{y(q_s) - M_2} \right\rangle_c^{(g)}$$

(1 − 4)

where one notes:

$$\mathrm{Tr}\, \frac{1}{\widehat{S}_i} = \mathrm{Tr}\, \left(\frac{1}{x(p_{1,1}) - M_1} \frac{1}{y(q_{1,1}) - M_2} \frac{1}{x(p_{1,2}) - M_1} \frac{1}{y(q_{1,2}) - M_2} \cdots \frac{1}{x(p_{1,k_1}) - M_1} \right) \tag{1-5}$$

and

$$\mathrm{Tr} \frac{1}{\widehat{S}_i} = \mathrm{Tr} \left(\frac{1}{x(p_{1,1}) - M_1} \frac{1}{y(q_{1,1}) - M_2} \frac{1}{x(p_{1,2}) - M_1} \frac{1}{y(q_{1,2}) - M_2} \cdots \frac{1}{x(p_{1,k_1}) - M_1} \frac{y(p_{1,1}) - M_2}{y(q_{1,k_1}) - M_2} \right). \tag{1-6}$$

These terms are monovalued functions as long as the p and q variables stay in their respective physical sheets. When q_{1,k_1} belongs to the y-physical sheet in the vincinity of ∞_y, all its images $\tilde{q}^j_{1,k_1}$ lie in the x-physical sheet in the vincinity of ∞_x. Thus, this expression vanishes for $p_{1,1} \to \tilde{q}^j_{1,k_1}$ [8]. Hence the Lagrange interpolation formula reads

$$U^{(g)}_{k_1,\dots,k_l;m;n}(x(p_{1,1})) = \sum_{j=1}^{d_1} \mathop{\mathrm{Res}}_{r \to \tilde{q}^j} \frac{H^{(0)}_{1;0;0}(p_{1,1}, q_{1,k_1}) U^{(g)}_{k_1,\dots,k_l;m;n}(x(r))(y(p_{1,1}) - y(q)) dx(r)}{(x(p_{1,1}) - x(r))(y(r) - y(q)) H^{(0)}_{1;0;0}(r, q_{1,k_1})}, \tag{1-7}$$

where we have defined:

$$U^{(g)}_{k_1,\dots,k_l;m;n}(x(p_{1,1})) := Pol_{x(p_{1,1})} V_1'(x(p_{1,1})) H^{(g)}_{k_1,\dots,k_l;m;n}(S_1, S_2, \dots, S_l; p_1, \dots, p_m; q_1, \dots, q_n). \tag{1-8}$$

Insert this formula into Eq. (1-3) and get:

$$H^{(g)}_{\mathbf{k_L};m;n}(\mathbf{S_L}; p_1, \dots, p_m; q_1, \dots, q_n) = \mathop{\mathrm{Res}}_{r \to p_{1,1}, \tilde{q}^j} \frac{H^{(0)}_{1;0;0}(p_{1,1}, q_{1,k_1})\ \mathrm{RHS}|_{p_{1,1} := r}}{(x(p_{1,1}) - x(r))(y(q) - y(r)) H^{(0)}_{1;0;0}(r, q_{1,k_1})} \tag{1-9}$$

where RHS denotes all the terms in the right hand side of Eq. (1-3).

One can simplify some of the terms by changing the integration contour. Indeed, consider any term of the form $d_{p_{i,\alpha}} \left(\frac{f(p_{i,\alpha})}{x(p_{1,1}) - x(p_{i,\alpha})} \right)$ in the RHS of Eq. (1-3), one can compute its contribution to the preceding formula:

$$\begin{aligned} & d_{p_{i,\alpha}}\ \mathrm{Res}_{\,r \to p_{1,1}, \tilde{q}^j} \frac{H^{(0)}_{1;0;0}(p_{1,1}, q_{1,k_1}) f(p_{i,\alpha})}{(x(p_{1,1}) - x(r))(y(q) - y(r))(x(p_{1,1}) - x(p_{i,\alpha})) H^{(0)}_{1;0;0}(r, q_{1,k_1})} = \\ & = d_{p_{i,\alpha}}\ \mathrm{Res}_{\,x(r) \to x(p_{1,1}), x(\tilde{q}^j)} \frac{H^{(0)}_{1;0;0}(p_{1,1}, q_{1,k_1}) f(p_{i,\alpha})}{(x(p_{1,1}) - x(r))(y(q) - y(r))(x(p_{1,1}) - x(p_{i,\alpha})) H^{(0)}_{1;0;0}(r, q_{1,k_1})} \end{aligned} \tag{1-10}$$

[8]This term does not vanish when $p_{1,1} \to q_{1,k_1}$ because of the discontinuity of these functions when $p_{1,1}$ changes x-sheets.

since one can check that the integrand is a polynomial in $x(r)$. We can now move the integration contour in the x basis and we get:

$$d_{p_{i,\alpha}} \operatorname*{Res}_{x(r)\to x(p_{i,\alpha})} \frac{H^{(0)}_{1;0;0}(p_{1,1},q_{1,k_1})f(p_{i,\alpha})}{(x(p_{1,1})-x(r))(y(q)-y(r))(x(p_{1,1})-x(p_{i,\alpha}))H^{(0)}_{1;0;0}(r,q_{1,k_1})}. \tag{1-11}$$

This residue can be evaluated by using one more time Eq. (1-3) and recalling that only the terms of the form $H_{0;2;0}(r,p_{i,\alpha})$ have such poles. This finally gives the result, i.e. eq.4-1:

$$\boxed{\begin{array}{l}
H^{(g)}_{\mathbf{k_L};m;n}(\mathbf{S_L};p_1,\dots,p_m;q_1,\dots,q_n) = \\
\operatorname*{Res}_{r\to p_{1,1},p_{i,\alpha},p_j,\tilde{q}^j_{1,k_1}} \frac{H^{(0)}_{1;0;0}(p_{1,1},q_{1,k_1})}{(x(p_{1,1})-x(r))(y(q_{1,k_1})-y(r))H^{(0)}_{1;0;0}(r,q_{1,k_1})} \\
\Big\{ \sum_{h=1}^{g} H^{(h)}_{0;1;0}(r)H^{(g-h)}_{k_1,\dots,k_l;m;n}(S_1(r),S_2,\dots,S_l;p_1,\dots,p_m;q_1,\dots,q_n) \\
+\sum_h \sum_{A\bigcup B=\{2,\dots,l\}} \sum_{I,J} H^{(h)}_{k_1,\mathbf{k_A};|I|;|J|}(S_1(r),\mathbf{S_A};\mathbf{p_I};\mathbf{q_J})H^{(g-h)}_{\mathbf{k_B};m-|I|+1;n-|J|}(\mathbf{S_B};r,\mathbf{p_{M/I}};\mathbf{q_N/J}) \\
+\sum_h \sum_{A\bigcup B=\{2,\dots,l\}} \sum_{\alpha=2}^{k_1}\sum_{I,J} H^{(h)}_{k_1-\alpha+1,\mathbf{k_B};m-|I|;n-|J|}(\{p_{1,\alpha},q_{1,\alpha},\dots p_{1,k_1},q_{1,k_1}\},\mathbf{S_B};\mathbf{p_{M/I}};\mathbf{q_{N/J}}) \\
\quad\times\frac{H^{(g-h)}_{\alpha-1,\mathbf{k_A};|I|;|J|}(\{r,q_{1,1},\dots p_{1,\alpha-1},q_{1,\alpha-1}\},\mathbf{S_A};\mathbf{p_I};\mathbf{q_J})}{x(p_{1,\alpha})-x(r)} \\
+\sum_{i=2}^{l}\sum_{\alpha=1}^{k_i} \frac{1}{x(p_{i,\alpha})-x(r)}\times \\
\quad\times H^{(g)}_{k_1+k_i,\mathbf{k_{L/\{1,i\}}};m;n}(\{S_1(r),p_{i,\alpha},q_{i,\alpha},p_{i,\alpha+1},\dots,q_{i,k_i},p_{i,1},\dots,p_{i,\alpha-1},q_{i,\alpha-1}\},\mathbf{S_{L/\{1,i\}}};\mathbf{p_M};\mathbf{q_N}) \\
+\sum_{\alpha=2}^{k_1}\frac{1}{x(p_{1,\alpha})-x(r)}\times \\
H^{(g-1)}_{\alpha-1,k_1-\alpha+1,\mathbf{k_{L/\{1\}}};m;n}(\{r,q_{1,1},\dots p_{1,\alpha-1},q_{1,\alpha-1}\},\{p_{1,\alpha},q_{1,\alpha},\dots p_{1,k_1},q_{1,k_1}\},\mathbf{S_{L/\{1\}}};\mathbf{p_M};\mathbf{q_N}) \\
+H^{(g-1)}_{\mathbf{k_L};m+1;n}(\mathbf{S_K}(r);r,\mathbf{p_M};\mathbf{q_N})\Big\}
\end{array}}$$

(1-12)

This recursion equation is a triangular system, thus it allows to compute any $H(S)$ recursively.

References

[1] J.Ambjørn, L.Chekhov, C.F.Kristjansen and Yu.Makeenko, “Matrix model calculations beyond the spherical limit”, *Nucl.Phys.* **B404** (1993) 127–172; Erratum ibid. **B449** (1995) 681, hep-th/9302014.

[2] J.-E. Bourgine, K. Hosomichi, I. Kostov, Y. Matsuo, ”Scattering of Long Folded Strings and Mixed Correlators in the Two-Matrix Model”, hep-th, arXiv:0709.3912v1.

[3] E. Brezin, C. Itzykson, G. Parisi, and J. Zuber, *Comm. Math. Phys.* **59**, 35 (1978).

[4] L.Chekhov, B.Eynard, N.Orantin, “Free energy topological expansion for the 2-matrix model”, *J. High Energy Phys.* **JHEP12**(2006)053, math-ph/0603003.

[5] J.M. Daul, V. Kazakov, I.K. Kostov, “Rational Theories of 2D Gravity from the Two-Matrix Model”, *Nucl. Phys.* **B409**, 311-338 (1993), hep-th/9303093.

[6] P. Di Francesco, P. Ginsparg, J. Zinn-Justin, “2D Gravity and Random Matrices”, *Phys. Rep.* **254**, 1 (1995).

[7] B. Eynard, ”Eigenvalue distribution of large random matrices, from one matrix to several coupled matrices”, Nuc. Phys. B506,3 633-664 (1997), cond-mat/9707005.

[8] B. Eynard, ”Large N expansion of the 2-matrix model”, JHEP 01 (2003) 051, hep-th/0210047.

[9] B. Eynard, ”Master loop equations, free energy and correlations for the chain of matrices”, JHEP11(2003)018, hep-th/0309036.

[10] B. Eynard, “Topological expansion for the 1-hermitian matrix model correlation functions”, JHEP/024A/0904, hep-th/0407261.

[11] B. Eynard, N. Orantin, ”Invariants of algebraic curves and topological expansion”, Communications in Number Theory and Physics, Vol 1, Number 2, math-ph/0702045, ccsd-hal-00130963.

[12] B.Eynard, N.Orantin, “Topological expansion of the 2-matrix model correlation functions: diagrammatic rules for a residue formula”, *J. High Energy Phys.* **JHEP12**(2005)034, math-ph/0504058.

[13] B.Eynard, “ Formal matrix integrals and combinatorics of maps”, math-ph/0611087.

[14] B.Eynard, N.Orantin, “Mixed correlation functions in the 2-matrix model, and the Bethe Ansatz”, *JHEP* 0508 028 (2005), math-ph/0504058.

[15] B.Eynard, N.Orantin, “Topological expansion of mixed correlations in the hermitian 2 Matrix Model and x-y symmetry of the F_g invariants”, arXiv:0705.0958.

[16] Y. Hikida, V. Schomerus, ”H_3^+ WZNW model from Liouville field theory”, arXiv:0706.1030.

[17] V.A. Kazakov, “Ising model on a dynamical planar random lattice: exact solution”, *Phys Lett.* **A119**, 140-144 (1986).

[18] A.A. Migdal, *Phys. Rep.* **102**(1983)199.

[19] S. Ribault, J. Teschner, "H(3)+ correlators from Liouville theory", hep-th arXiv:hep-th/0502048.

[20] M. Staudacher, " Combinatorial solution of the 2-matrix model", *Phys. Lett.* **B305** (1993) 332-338.